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Authors

Bander, M
Coulter, PW
Shaw, GL

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NONEQUIVALENCE OF THE ONE-CHANNEL N/D EQUATIONS
WITH INELASTIC UNITARITY AND THE MULTICHANNEL ND^{-1} EQUATIONS

Myron Bander*

Stanford Linear Accelerator Center, Stanford University, Stanford, California

and

Philip W. Coulter† and Gordon L. Shaw†

Institute of Theoretical Physics, Department of Physics, Stanford University, Stanford, California

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Consider a partial-wave elastic-scattering amplitude¹ for two spinless particles of equal mass, M , as a function of $s = 4(k^2 + M^2)$:

$$A = \frac{1}{2i\rho}(S-1) = \frac{1}{2i\rho}(\eta e^{2i\delta} - 1) = B + {}^RA, \quad (1)$$

where ρ is a kinematical factor and the “generalized potential” B is regular in the physical region, whereas RA has cuts only for $s > 4M^2 \equiv s_E$. The inelastic partial-wave cross section σ_r^l is determined by η alone:

$$\sigma_r^l = \pi k^2 (2l+1)(1-\eta^2). \quad (2)$$

Given B and η , we can determine $A \equiv N/D$ using

the Frye-Warnock equations^{2,3}

$$\frac{2\eta(s)}{1+\eta(s)} \text{Re}N(s) = \bar{B}(s) + \frac{1}{\pi} \int_{s_E}^{\infty} \frac{[\bar{B}(s') - \bar{B}(s)] 2\rho(s') \text{Re}N(s') ds'}{(s'-s)[1+\eta(s')]},$$

$$\bar{B}(s) = B(s) + \frac{P}{\pi} \int_{s_I}^{\infty} \frac{[1-\eta(s')] ds'}{2\rho(s')(s'-s)},$$

$$D(s) = 1 - \frac{P}{\pi} \int_{s_E}^{\infty} \frac{2\rho(s') \text{Re}N(s') ds'}{(s'-s)[1+\eta(s')]} - i \frac{2\rho(s)}{1+\eta(s)} \text{Re}N(s) \theta(s-s_E),$$

$$\text{Im}N(s) = \frac{1-\eta(s)}{2\rho(s)} \text{Re}D(s) \theta(s-s_I) \quad (3)$$

[in addition to the usual left-hand cut in $N(s)$], where s_I is the lowest inelastic threshold. On the other hand, consider a set of coupled two-body channels with potentials B_{ij} . The amplitudes

$$A_{ij} = (S_{ij} - \delta_{ij}) \frac{1}{2i(\rho_i \rho_j)^{1/2}}$$

may be determined by the multichannel ND^{-1} formalism from the B_{ij} . Now take B_{11} and η determined from the $|A_{ij}|^2$ and calculate A from (3).⁴

The purpose of this note is to demonstrate by a simple example that the solution A is not in general equal to A_{11} .⁵ We find in our simple two-channel example described below that a sufficient condition for the two amplitudes A [calculated from (3)] and A_{11} (calculated by the multichannel ND^{-1} equations) to be identical is that the diagonal forces B_{22} are not strong enough to produce bound states in channel 2 in the absence of coupling between the channels. As one increases the strengths for the B_{22} beyond the value necessary to produce binding, complex-conjugate pairs of zeros in S_{11} move onto the physical sheet through the inelastic cut ($s > s_I$). The two calculations then disagree. Thus the physical situation in which we have a B_{22} strong enough to produce a bound state in channel 2 and then weakly couple it to the open channel 1 to produce a narrow resonance in A_{11} cannot be reproduced in the one-channel calculation (3). In addition, we demonstrate that, in our simple example, there are no poles of the S matrix on the physical sheet for complex values of s . (There did occur "ghost" poles on the negative real axis for some values of the B_{ij} , in which case A_{11} and A disagree. However, we disregard these unphysical situations.)

In order to carry out a substantial amount of the calculations analytically, we consider a two-channel nonrelativistic s -wave [$\rho_i = (s - s_i)^{1/2}$] system with the input (symmetric) B given by a single pole

$$B_{ij} = g_{ij} / (s + m). \quad (4)$$

Then

$$\begin{aligned} A_{ij} &= g_{ik} (D^{-1})_{kj} / s + m, \\ \frac{1}{4}(1 - \eta^2) &= \rho_1 \rho_2 |A_{12}|^2 \theta(s - s_2), \\ D_{ij} &= \delta_{ij} - g_{ij} \varphi_i, \\ \varphi_i &= -\frac{1}{2(s_i + m)^{1/2}} + \frac{(s_i + m)^{1/2}}{s + m} - \frac{(s_i - s)^{1/2}}{s + m}. \end{aligned} \quad (5)$$

The procedure is as follows: For given g_{ij} and m , calculate A_{11} and η from (5). Then using B_{11} and η as input we calculate A from (3) and compare it with A_{11} . [The integral equation (3) for $\text{Re}N(s)$ is solved numerically by the matrix inversion technique.] The next step in the program is to locate the zeros and poles of $S_{11} = 2i\rho_1 A_{11} + 1$. This problem is easily reduced to solving a quartic equation in the variable $(s - s_2)^{1/2}$; the same equation gives both zeros and poles of S_{11} as a function of the three g_{ij} 's for a given input pole position m . After solving for the roots, we determine whether they correspond to poles or zeros of S_{11} on the physical sheet [where $\text{Im}(s - s_2)^{1/2} \geq 0$ and $\text{Im}(s - s_1)^{1/2} \geq 0$] by putting these values back into the expression for S_{11} . We find that there are no poles in S_{11} on the physical sheet for complex values of s .

Now for given g_{11} and g_{12} , take g_{22} small; then we find from our numerical calculations that A_{11} agrees with A as calculated from (3). Increase g_{22} : For all $g_{22} > \text{some value } \bar{g}_{22}(g_{11}, g_{12}, m) > 2(s_2 + m)^{1/2}$ (the value for which channel 2 in the absence of coupling to channel 1 develops a bound state), the two amplitudes A_{11} and A disagree. Returning to the location of the zeros in S_{11} , we find that \bar{g}_{22} corresponds to the value for which a (double) zero in S_{11} occurs along the real axis above the inelastic threshold, i.e., η for some $s > s_2$ is equal to zero. We see for this situation that the integral equation (3) for $\text{Re}N$ is no longer Fredholm. For $g_{22} > \bar{g}_{22}(g_{11}, g_{12}, m)$, a pair of zeros in S_{11} (at complex-conjugate points) move from the real axis onto the physical sheet.

We investigated in great detail the case $g_{11} = 0$, i.e., no left-hand cut in channel 1. In this case the Ball-Frazer⁶ representation is applicable: We write a dispersion relation for the phase shift in channel 1 (which is valid when the S matrix has no zeros on the physical sheet):

$$\delta = -(s - s_1)^{1/2} \frac{P}{2\pi} \int_{s_2}^{\infty} \frac{\ln \eta(s') ds'}{(s' - s_1)^{1/2} (s' - s)}. \quad (6)$$

In addition, we note that the quartic equation for the zeros in S_{11} reduces to a cubic. We find that in all cases ($g_{11} = 0$) that both one-channel calculations (3) and (6) for A agree. They both break down and disagree⁷ with the two-channel A_{11} when zeros in S_{11} appear on the physical sheet, coming through the inelastic cut. It is clear that A as calculated from (6)

will disagree with A_{11} then, since zeros in S_{11} amount to cuts in δ which are not taken into account by (6).

The appearance of zeros (at α and α^*) of S_{11} on the physical sheet through the inelastic cut will also cause the Froissart⁸ one-channel N/D formalism to disagree with A_{11} . He introduces

$$R = \exp \left[-\frac{i(s-s_1)^{1/2}}{\pi} \int_{s_I}^{\infty} \frac{ds' \ln \eta(s')}{(s'-s_1)^{1/2}(s'-s)} \right],$$

and notes that $R^{-1}S$ satisfies elastic unitarity. However, R is not unique since we could multiply it by the factor

$$G = \frac{[\alpha - i(s-s_1)^{1/2}][\alpha^* - i(s-s_1)^{1/2}]}{[\alpha + i(s-s_1)^{1/2}][\alpha^* + i(s-s_1)^{1/2}]}.$$

This would presumably bring the one-channel calculation in agreement with the multichannel one. The G factor is clearly related to specifying the Castillejo-Dalitz-Dyson ambiguity.⁹

In summary, we found in our simple example that a sufficient condition for the one-channel calculation (3) to agree with the multichannel amplitude A_{11} is that the diagonal forces in the channels not explicitly considered should not be strong enough to produce bound states in the absence of coupling to channel 1.¹⁰ We speculate that this condition will hold in general.

It would be of interest to investigate more complicated examples, in particular, more complicated input "potentials" B and systems with more than two channels.

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¹We use units $\hbar = c = 1$.

²G. Frye and R. Warnock, Phys. Rev. **130**, 478 (1963).

³P. Coulter, A. Scotti, and G. Shaw, Phys. Rev. **136**, B1399 (1964).

⁴We consider the situation in which all the channels i have thresholds s_i greater than that of channel 1.

⁵A similar problem has been considered from a different approach by G. Chew (private communication) and E. Squires (to be published).

⁶J. Ball and W. Frazer, Phys. Rev. Letters **7**, 204 (1961).

⁷Note that a numerical comparison between A_{11} and A as calculated by (6) can be made to more significant figures than the calculation (3) since (3) involves the solution of an integral equation in addition to numerical integration.

⁸M. Froissart, Nuovo Cimento **22**, 191 (1961).

⁹L. Castillejo, R. Dalitz, and F. Dyson, Phys. Rev. **101**, 453 (1956).

¹⁰Thus, for example, it is felt that the conclusions of reference 3 are valid.